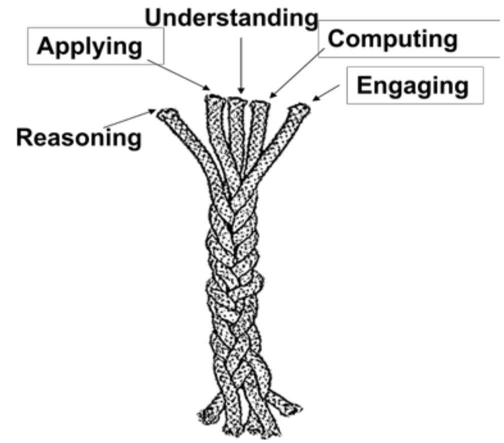
Mathematical proficiency has five strands:



(1) **Understanding:** Comprehending mathematical concepts, operations, and relations—knowing what mathematical symbols, diagrams, and procedures mean

(2) **Computing:** Carrying out mathematical procedures, such as adding, subtracting, multiplying, and dividing numbers flexibly, accurately, efficiently, and appropriately

(3) **Applying:** Being able to mathematically formulate problems and devise strategies for solving them using appropriate concepts and procedures

(4) **Reasoning:** Using logic to explain and justify a solution to a problem or to extend from something known to something not yet known

(5) **Engaging:** Seeing mathematics as sensible, useful, and doable—*if* you work at it—and being willing to do the work

The most important feature of mathematical proficiency is that these five strands are interwoven and interdependent. Other views of mathematics learning tend to emphasize only one aspect of proficiency, with the expectation that other aspects will develop as a consequence. For example, some people who have emphasized the need for students to master computations have assumed that understanding would follow. Others, focusing on students' understanding of concepts, have assumed that skill would develop naturally. By using these five strands, we have attempted to give a more rounded portrayal of successful mathematics learning.

National Research Council. (2002). Helping Children Learn Mathematics. pp. 8-10

Understanding: Comprehending mathematical concepts, operations, and relations—knowing what mathematical symbols, diagrams, and procedures mean

Understanding refers to students' grasp of fundamental mathematical ideas. Students with understanding know more than isolated facts and procedures. They know why a mathematical idea is important and the contexts in which it is useful. Furthermore, they are aware of many connections between mathematical ideas. In fact, the degree of students' understanding is related to the richness and extent of the connections they have made.

Students who learn with understanding have less to learn because they see common patterns in superficially different situations. If they understand the general principle that the order in which two numbers are multiplied doesn't matter—3×5 is the same as 5×3, for example—they have about half as many "number facts" to learn. Or if students understand the general principle that multiplying the dimensions of a three-dimensional object by a factor n increases its volume by the factor n^3, they can understand many situations in which objects of all shapes are proportionally expanded or shrunk. (They can understand, for example, why a 16-ounce cup that has the same shape as an 8-ounce cup is much less than twice as tall.)

Knowledge learned with understanding provides a foundation for remembering or reconstructing mathematical facts and methods, for solving new and unfamiliar problems, and for generating new knowledge. For example, students who thoroughly understand whole number operations can extend these concepts and procedures to operations involving decimals.

National Research Council. (2002). Helping Children Learn Mathematics. pp. 10-11

Computing: Carrying out mathematical procedures, such as adding, subtracting, multiplying, and dividing numbers flexibly, accurately, efficiently, and appropriately

Computing includes being fluent with procedures for adding, subtracting, multiplying, and dividing mentally or with paper and pencil and knowing when and how to appropriately use these procedures. Although the word *computing* implies arithmetic procedure, in this document it also refers to being fluent with procedures from other branches of mathematics, such as measurement (measuring lengths), algebra (solving equations), geometry (constructing similar figures), and statistics (graphing data). Being *fluent* means having the skill to perform the procedure efficiently, accurately, and flexibly.

Students need to compute basic number combinations (6 + 7, 17 - 9, 8 × 4, and so on) rapidly and accurately. They also need to become accurate and efficient with algorithms—step-by-step procedures for adding, subtracting, multiplying, and dividing multi-digit whole numbers, fractions, and decimals—and for doing other computations. For example, all students should have an algorithm for multiplying 64 and 37 they understand that is reasonably efficient and general enough to be used with other two-digit numbers, and that can be extended to use with larger numbers.

Accuracy and efficiency with procedures are important, but computing also supports understanding. By working through procedures that are general enough for solving a whole class of problems, such as a procedure for adding any two fractions, students gain appreciation for the fact that mathematics is predictable, well structured, and filled with patterns.

Developing computational skills and developing understanding are often seen as competing for attention in school mathematics. But, skill against understanding creates a false dichotomy. Understanding makes it easier to learn skills, while learning procedures can strengthen and develop mathematical understanding.

National Research Council. (2002). *Helping Children Learn Mathematics. pp. 11, 13*

To many students, practice is as much a part of studying mathematics as of playing a sport or a musical instrument. The role of practice in mathematics, as in sports or music, is to be able to execute procedures automatically without conscious thought. A procedure is practiced over and over until so-called *automaticity* is attained.

There are cognitive benefits to automatization. The more automatically a procedure can be executed, the less mental effort is required. Since each person has a limited amount of mental effort that he or she can expend at any one time, more complex tasks can be done well only when some of the subtasks are automatic. Hence, the automatization of mathematical procedures is justifiable when those procedures are regularly required to complete other tasks. For example, basic multiplication combinations such as $4 \times 6 = 24$ and $6 \times 7 = 42$ are needed for estimation, multi-digit multiplication, single-digit division, multi-digit division, and addition and multiplication of fractions, to name a few. Therefore, multiplication combinations need to be practiced until they can be produced quickly and effortlessly. The availability of calculators and computers raises the question of which mathematical procedures today need to be practiced to the point of automatization. Single-digit whole number addition, subtraction, multiplication, and division certainly need to be automatic, since they are used in almost all other numerical procedures. Opinions vary, however, as to which other procedures should be made automatic.

National Research Council. (2001). *Adding It Up: Helping Children Learn Mathematics. p. 351*

Fluency demands more of students than memorizing a single procedure. Fluency rests on a well-built mathematical foundation with three parts:

1. An understanding of the meaning of the operations and their relationships to each other—for example, the inverse relationship between multiplication and division

2. The knowledge of a large repertoire of number relationships, including addition and multiplication “facts,” as well as other relationships such as how 4 × 5 is related to 4 × 50

3. A thorough understanding of the base-ten number system, how numbers are structured in this system, and how the place value system of numbers behaves in different operations—for example, that 24 + 10 = 34 or that 24 × 10 = 240

- Computational fluency is an essential goal for school mathematics.
- The methods that a student uses to compute should be grounded in understanding.
- Students can achieve computational fluency using a variety of methods and should, in fact, be comfortable with more than one approach.
- Students should have opportunities to invent strategies for computing using their knowledge of place value, properties of numbers, and operations.

Russell, S. J. (2000). *Teaching Children Mathematics, 7*(3), pp. 154-156.